Unimodality of generalized Gaussian coefficients.

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Abstract

A combinatorial proof of the unimodality of the generalized q-Gaussian coefficients $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ based on the explicit formula for Kostka-Foulkes polynomials is given.

1^0. Let us mention that the proof of the unimodality of the generalized Gaussian coefficients based on theoretic-representation considerations was given by E.B. Dynkin [1] (see also [2], [10], [11]). Recently K.O'Hara [6] gave a constructive proof of the unimodality of the Gaussian coefficient $\begin{bmatrix} n+k \\ n \end{bmatrix}_q = s_{(k)}(1, \cdots, q^k)$, and D. Zeilberger [12] derived some identity which may be consider as an "algebraization" of O'Hara's construction. By induction this identity immediately implies the unimodality of $\begin{bmatrix} n+k \\ n \end{bmatrix}_q$. Using the observation (see Lemma 1) that the generalized Gaussian coefficient $\begin{bmatrix} n \\ \lambda' \end{bmatrix}_q$ may be identified (up to degree q) with the Kostka-Foulkes polynomial $K_{\widetilde{\lambda,\mu}}(q)$ (see Lemma 1), the proof of the unimodality of $\begin{bmatrix} n \\ \lambda' \end{bmatrix}_q$ is a simple consequence of the exact formula for Kostka-Foulkes polynomials contained in [4]. Furthermore the expression for $K_{\widetilde{\lambda,\mu}}(q)$ in the case $\lambda = (k)$ coincides with identity (KOH) from [8]. So we obtain a generalization and a combinatorial proof of (KOH) for arbitrary partition λ.

2^0. Let us recall some well known facts which will be used later. We base ourselves [9] and [5]. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition, $|\lambda|$ be the sum of its parts λ_i, $n(\lambda) = \sum_i (i-1)\lambda_i$ and $\left[\begin{array}{c} n \\ \lambda \end{array} \right]_q$ be the generalized Gaussian coefficient.

Recall that

$$s_\lambda(1, \cdots, q^n) = q^{n(\lambda)} \left[\begin{array}{c} n \\ \lambda' \end{array} \right]_q = \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}}. \qquad (1)$$

Here $c(x)$ is the content and $h(x)$ is the hook length corresponding to the box $x \in \lambda$, [5].

Lemma 1 *Let λ be a partition and fix a positive integer n. Consider new partitions $\widetilde{\lambda} = (n \cdot |\lambda|, \lambda)$ and $\mu = (|\lambda|^{n+1})$. Then*

$$q^{\frac{n(n-1)}{2} \cdot |\lambda| + n(\lambda)} \left[\begin{array}{c} n \\ \lambda' \end{array} \right]_q = K_{\widetilde{\lambda}, \mu}(q). \qquad (2)$$

Proof. We use the description of the polynomial $q^{|\lambda| + n(\lambda)} \cdot \left[\begin{array}{c} n \\ \lambda' \end{array} \right]_q$ as a generating function for the standard Young tableaux of the shape λ filled with numbers from the interval $[1, \cdots, n]$. Let us denote this set of Young tableaux by $STY(\lambda, \leq n)$. Then

$$q^{|\lambda| + n(\lambda)} \left[\begin{array}{c} n \\ \lambda' \end{array} \right]_q = \sum_{T \in STY(\lambda, \leq n)} q^{|T|} \ . \qquad (3)$$

Here $|T|$ is the sum of all numbers filling the boxes of T . For any tableau T (or diagram λ) let us denote by $T[k]$ (or $\lambda[k]$) the part of T (or λ) consisting of rows starting from the $(k+1)$-st one. Given tableau $T \in STY(\lambda, \leq n)$, then consider tableau $\widetilde{T} \in STY(\widetilde{\lambda}, \mu)$ such that $\widetilde{T}[1] = T + supp\ \lambda[1]$, and we fill the first row of $\tilde{T}$ with all remaining numbers in increasing order from left to right. Here for any diagram λ we denote by $supp\lambda$ the plane partition of the shape λ and content $(1^{|\lambda|})$. It is easy to see that

$$c(\widetilde{T}) = |T| + \frac{(n+1)(n-2)}{2} \cdot |\lambda| \ ,$$

so we obtain the identity (2). ∎

Let us consider an explanatory example. Assume $\lambda = (2,1)$, $n = 3$. Then $\widetilde{\lambda} = (9,2,1)$, $\mu = (3^4)$. It is easy to see that $|STY(\lambda, \le 3)| = 8$.

T	$\lvert T\rvert$	$\widetilde{T}$	$c(\widetilde{T})$
$\begin{array}{ll} 1 & 1 \\ 2 & \end{array}$	4	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 2 \\ 3 \end{array}$	10
$\begin{array}{ll} 1 & 1 \\ 3 & \end{array}$	5	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 \\ 2 & 2 \\ 4 \end{array}$	11
$\begin{array}{ll} 1 & 2 \\ 2 & \end{array}$	5	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 \\ 2 & 3 \\ 3 \end{array}$	11
$\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$	6	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 3 \\ 4 \end{array}$	12
$\begin{array}{ll} 1 & 3 \\ 2 & \end{array}$	6	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 4 \\ 3 \end{array}$	12
$\begin{array}{ll} 1 & 3 \\ 3 & \end{array}$	7	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 4 \\ 4 \end{array}$	13
$\begin{array}{ll} 2 & 2 \\ 3 & \end{array}$	7	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 4 \\ 3 & 3 \\ 4 \end{array}$	13
$\begin{array}{ll} 2 & 3 \\ 3 & \end{array}$	8	$\begin{array}{lllllllll} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 4 \\ 4 \end{array}$	14

Now we would like to use the formula for Kostka-Foulkes polynomials, obtained in [4].

$\mathbf{3^0}$. First let us recall some definitions from [4]. Given a partition λ and composition μ, a configuration $\{\nu\}$ of the type (λ,μ) is, by definition, a collection of partitions $\nu^{(1)},\nu^{(2)},\cdots$ such that

1) $|\nu^{(k)}|=\sum_{j\ge k+1}\lambda_j$;
2) $P_n^{(k)}(\lambda,\mu):=Q_n(\nu^{(k-1)})-2Q_n(\nu^{(k)})+Q_n(\nu^{(k+1)})\ge 0$ for all $k,n\ge 1$, where $Q_n(\lambda):=\sum_{j\le n}\lambda'_j$,and $\nu^{(0)}=\mu$.

Proposition 1 *[4] Let λ be a partition and μ be a composition, then*

$$K_{\lambda,\mu}(q)=\sum_{\{\nu\}}q^{c(\nu)}\prod_{k,n}\left[\begin{array}{c}P_n^{(k)}(\lambda,\mu)+m_n(\nu^{(k)})\\ m_n(\nu^{(k)})\end{array}\right]_q, \tag{4}$$

where the summation in (4) is taken over all configurations of $\{\nu\}$ of the type (λ,μ), $m_n(\nu^{(k)})=(\nu^{(k)})'_n-(\nu^{(k)})'_{n+1}$.

From Lemma 1 and Proposition 1 we deduce

Theorem 1 *Let λ be a partition. Then*

$$\left[\begin{array}{c}N\\ \lambda'\end{array}\right]_q=\sum_{\{\nu\}}q^{c_0(\nu)}\prod_{k,n}\left[\begin{array}{c}P_n^{(k)}(\lambda,N)+m_n(\nu^{(k)})\\ m_n(\nu^{(k)})\end{array}\right]_q, \tag{5}$$

where the summation in (5) is taken over all collections $\{\nu\}$ of partitions $\{\nu\}=\{\nu^{(1)},\nu^{(2)},\cdots\}$ such that

1) $|\nu^{(k)}|=\sum_{j\ge k}\lambda_j$, $k\ge 1$, $|\nu^{(0)}|=0$,

2) $P_n^{(k)}(\nu,N):=n(N+1)\cdot\delta_{k,1}+Q_n(\nu^{(k-1)})-2Q_n(\nu^{(k)})+Q_n(\nu^{(k+1)})\ge 0$, for all $k,n\ge 1$. Here

$$c_0(\nu)=n(\nu^{(1)})-n(\lambda)+\sum_{k,n\ge 1}\left(\begin{array}{c}\alpha_n^{(k)}-\alpha_n^{(k+1)}\\ 2\end{array}\right),\ \ \alpha_n^{(k)}:=(\nu^{(k)})'_n \tag{6}$$

and by definition $\left(\begin{array}{c}\alpha\\ 2\end{array}\right):=\frac{\alpha(\alpha-1)}{2}$ *for any* $\alpha\in\mathbf{R}$.

The identity (5) may be consided as a generalization of the (KOH) - identity (see [8]) for arbitrary partition λ.

Corollary 1 *The generalized q-Gaussian coefficient* $\left[\begin{array}{c}n\\ \lambda'\end{array}\right]_q$ *is a symmetric and unimodal polynomial of degree* $(N-1)|\lambda|-n(\lambda)$.

Proof. First, it is well known (e.g. [10],[11]) that the product of symmetric and unimodal polynomials is again symmetric and unimodal. Secondly, we use a well known fact (e.g.[10]), that the ordinary q-Gaussian coefficient $\left[\begin{matrix} m+n \\ n \end{matrix}\right]_q$ is a symmetric and unimodal polynomial of degree mn. So in order to prove Corollary 1, it is sufficient to show that the sum

$$2c_0(\nu) + \sum_{k,n} m_n(\nu^{(k)}) P_n^{(k)}(\nu, N) \tag{7}$$

is the same for all collections of partitions $\{\nu\}$ which satisfy the conditions 1) and 2) of the Theorem 1. In oder to compute the sum (7), we use the following result (see [4]):

Lemma 2 *Assume* $\{\nu\}$ *to be a configuration of the type* (λ, ν). *Then*

$$\sum_{k,n} m_n(\nu^{(k)}) P_n^{(k)}(\nu, \mu) = 2n(\mu) - 2c(\nu) - \sum_{n \geq 1} \mu'_n \cdot \alpha_n^{(1)}. \tag{8}$$

Using Lemma 2, it is easy to see that the sum (7) is equal to $(N-1)|\lambda| - n(\lambda)$. This concludes the proof. ■

Note that in the proof of Corollary 1 we use symmetry and unimodality of the ordinary q-Gaussian coefficient $\left[\begin{matrix} m+n \\ n \end{matrix}\right]_q$. However, we may prove the unimodality of $\left[\begin{matrix} m+n \\ n \end{matrix}\right]_q$ by induction using the identity (5) for the case $\lambda = (1^n)$, $N = m$.

Remark 1. The unimodality of generalized q-Gaussian coefficients was also proved in the recent preprint [7]. The proof in [7] uses the result from [4]. However [7] does not contain the identity (5).

Remark 2. The proof of the identity (4) given in [4] is based on the construction and properties of the bijection (see [4])

$$STY(\lambda, \mu) \rightleftharpoons QM(\lambda, \mu).$$

It is an interesting task to obtain an analytical proof of (5). In the case $q = 1$ such a proof was obtained in [3].

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